Conserved current for the Cotton tensor, black hole entropy and equivariant Pontryagin forms

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Abstract

The Chern-Simons lagrangian density in the space of metrics of a 3-dimensional manifold M is not invariant under the action of diffeomorphisms on M. However, its Euler-Lagrange operator can be identified with the Cotton tensor, which is invariant under diffeomorphims. As the lagrangian is not invariant, Noether Theorem cannot be applied to obtain conserved currents. We show that it is possible to obtain an equivariant conserved current for the Cotton tensor by using the first equivariant Pontryagin form on the bundle of metrics. Finally we define a hamiltonian current which gives the contribution of the Chern-Simons term to the black hole entropy, energy and angular momentum.

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1 Introduction

In Topologically Massive Gravity the lagrangian is given by the Hilbert Einstein lagrangian plus a Chern-Simons term (e.g. see [1]). In dimension 3, although the Chern-Simons lagrangian for metrics is not invariant under the action of the group of diffeomorphisms on the manifold, its the Euler-Lagrange operator can be identified with the Cotton tensor which is invariant. In fact, the Cotton tensor does not admit a diffemorphisms invariant lagrangian (see [2] and references therein for the properties and history of the Cotton tensor).

Since the Chern-Simons lagrangian is not diffeomorphisms invariant, we cannot apply Noether theorem in order to obtain the corresponding conserved currents. This note aims to show that it is possible to define an equivariant conserved current. We show in Section 4 that this current appears in a natural way from the geometry of the jet bundle of metrics and that it is provided

by the equivariant Pontryagin forms defined in [3]. As the conserved currents associated to invariance under diffeomorphisms are globally exact on shell we have $J_{CS} \approx dQ_{CS}(X)$, where Q_{CS} is the Noether charge. We show that Q_{CS} is given by the Schouten tensor of the metric. Finally in Section 5 we follow Wald's Noether method to compute the contribution of the Chern-Simons term to the black hole entropy. As the Chern-Simons lagrangian is not invariant, the current Q_{CS} does not give the correct value of the entropy. We define a Hamiltonian current q_{CS} for the Chern-Simons term by adding to the Noether charge Q_{CS} an additional term which is also obtained from the equivariant Pontryagin form. We show that this hamiltonian current coincides with a current defined in [4] for constant vector fields, and hence gives the same value for the contribution of the Chern-Simons term to the black-hole energy and angular momentum. Moreover, we show that the current q_{CS} also gives the correct value of the contribution of the Chern-Simons term to the black-hole entropy computed in [5].

We use the approach to the calculus of variations in terms of differential forms on jet bundles. In Section 3 we recall the basics results on the geometry of the jet bundle of the bundle of metrics $J\mathcal{M}_M$, and we define the Pontryagin forms, equivariant Pontryagin forms and Chern-Simons forms on $J\mathcal{M}_M$.

Most of our results are based on very general properties of the geometry of the jet bundle that can be easily generalized to higher dimensions. However the final result for the black hole entropy is not so easily obtained in higher dimensions and for this reason we consider only the 3-dimensional case.

In the following, the word metric means Riemannian or pseudo-Riemannian metric.

2 The Cotton tensor and the Chern-Simons lagrangian

Let us recall the computation of the Euler Lagrange operator of the gravitational Chern-Simons lagrangian. We follow the exposition in [6].

In dimension 3 the Chern-Simons lagrangian for metrics on a 3-manifold M is given locally by

$$\lambda_{CS} = \alpha \mathrm{tr} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right), \tag{1}$$

(in Section 3 we give a definition of Chern-Simons lagrangian valid globally).

If we consider an arbitrary variation of the metric δg_{ab}, then the variation of λ_{CS} is given by

$$\delta\lambda_{CS} = 2\alpha \mathrm{tr} \left(\delta\Gamma \wedge R \right) + \alpha d \left(\delta\Gamma \wedge \Gamma \right) \tag{2}$$

Using the expression for the variation of the Christoffel symbols

$$\delta\Gamma^a_{bc} = \frac{1}{2} g^{aj} \left(\nabla_b \delta g_{jc} + \nabla_c \delta g_{jb} - \nabla_j \delta g_{bc} \right)$$

and the expression for the Riemann tensor in terms of the Ricci tensor in dimension 3

$$R^a_{bcd} = \delta^a_c \left(R_{bd} - \frac{1}{2} g_{bd} R \right) - \delta^a_d \left(R_{bc} - \frac{1}{2} g_{bc} R \right) + g_{bd} R^a_c - g_{bc} R^a_d \tag{3}$$

we obtain

$$\mathrm{tr}\left(\delta\Gamma \wedge R\right) = \nabla_b \delta g_{ia} R^i_c dx^a \wedge dx^b \wedge dx^c$$

If we integrate by parts it follows that

$$\begin{aligned} \mathrm{tr}\left(\delta\Gamma \wedge R\right) &= \left(\partial_b \left(\delta g_{ia} R^i_c\right) - \delta g_{ia} \left(\nabla_b R^i_c\right)\right) dx^a \wedge dx^b \wedge dx^c \\ &= -\left(d\left(\delta g_{ia} R^i_c dx^a \wedge dx^c\right) + \delta g_{ia} \left(\nabla_b R^i_c\right) dx^a \wedge dx^b \wedge dx^c\right). \end{aligned}$$

The first term is an exact form, and the second one can be expressed in terms of the Cotton tensor

$$C^{ab} = -\frac{1}{2\sqrt{-|g|}} \left(\varepsilon^{ija} \nabla_i R^b_j + \varepsilon^{ijb} \nabla_i R^a_j\right)$$

and we obtain

$$\mathrm{tr}\left(\delta\Gamma \wedge R\right) = \delta g_{ij} C^{ij} \mathrm{vol} + dN(\delta g) \tag{4}$$

where $\mathrm{vol} = \sqrt{|g|} d^3x$ and $N(\delta g) = -\delta g_{ia} R^i_c dx^a \wedge dx^c$.

By replacing (4) on (2) we obtain the first variational formula

$$\delta\lambda_{CS} = 2\alpha \delta g_{ij} C^{ij} \mathrm{vol} + \alpha d(N(\delta g) + \mathrm{tr}(\delta\Gamma \wedge \Gamma)) \tag{5}$$

Hence the Euler-Lagrange operator of λ_{CS} can be identified with the Cotton tensor.

Now we consider the natural action of the diffeomorphism group of M in the space of metrics on M. Let us consider a variation of the metric $\delta_X g = -L_X g$ induced by an infinitesimal diffeomorphism $X \in \mathfrak{X}(M)$. If in local coordinates we have $X = X^i \partial/\partial x^i$ then

$$\delta_X g_{ab} = -\left(\partial_k g_{ab} X^k + g_{kb} \partial_a X^k + g_{ak} \partial_b X^k\right) = -\left(g_{ja} \nabla_b X^j + g_{jb} \nabla_a X^j\right) \tag{6}$$

By replacing (6) on (5) we obtain

$$\delta_X \lambda_{CS} = 2\alpha \delta_X g_{ab} C^{ab} \mathrm{vol} + \alpha d(N(\delta_X g) + \mathrm{tr}(\delta_X \Gamma \wedge \Gamma)). \tag{7}$$

It can be seen (see Section 3.4) that we have $\delta_X \lambda_{CS} = d\sigma$ for certain form σ, and hence $J_{CS} = N(\delta_X g) + \mathrm{tr}(\delta_X \Gamma \wedge \Gamma) - \sigma$ is a conserved current, i.e. $dJ_{CS} = 2\alpha \delta_X g_{ab} C^{ab} \mathrm{vol} \approx 0$. We show in Section 4 that this conserved current can be obtained directly by combining equation (4) with the equivariant Pontryagin forms.

3 Equivariant Pontryagin forms in the jet bundle of metrics

3.1 The jet bundle of the bundle of metrics

Let us recall some results on the geometry of the jet bundle of the bundle of metrics and the relation with the concepts introduced in the previous section. We refer to [7, 8] for more details on the description of variational calculus in terms of the geometry of jet bundles and to ([9, 3]) for the geometry of the bundle of metrics.

We denote by $J\mathcal{M}_M$ the jet bundle of the bundle of metrics i.e., $J\mathcal{M}_M$ is the space obtained when the derivatives of the metric are considered as independent variables. Hence if (x^i) are coordinates on M, then the coordinates on $J\mathcal{M}_M$ are $(x^i, g_{ij}, g_{ij,k}, g_{ij,kr}, \ldots)$.

Sometimes it is interesting to consider $J\mathcal{M}_M$ as an infinitesimal version of the product $M \times \mathrm{Met}M$, where $\mathrm{Met}M$ is the space of metrics on M. Both spaces are related by the evaluation map $M \times \mathrm{Met}M \to J\mathcal{M}_M$, $(x, g) \mapsto (x^i, g_{ij}(x), \partial_k g_{ij}(x), \ldots)$. For example, we have a canonical decomposition $T^*J\mathcal{M}_M \cong T^*M \oplus V^*(J\mathcal{M}_M)$, where $V^*(J\mathcal{M}_M)$ is the space contact (or vertical) 1-forms generated by the 1-forms $\delta g_{ij,I} = Dg_{ij,I} - g_{ij,I+k}Dx^k$, and D denotes the exterior differential on $\Omega^r(J\mathcal{M}_M)$. Accordingly we have

$$\Omega^r(J\mathcal{M}_M) = \oplus_{p+q=r}\Omega^{p,q}(J\mathcal{M}_M), \tag{8}$$

where $\Omega^{p,q}(J\mathcal{M}_M)$ is the space of p-horizontal and q-vertical forms. The exterior differential D on $\Omega^r(J\mathcal{M}_M)$ splits into horizontal and vertical differentials $D = d + \delta$, where the horizontal differential $d\colon \Omega^{p,q}(J\mathcal{M}_M) \to \Omega^{p+1,q}(J\mathcal{M}_M)$ measures the changes on M, whereas the vertical differential $\delta\colon \Omega^{p,q}(J\mathcal{M}_M) \to \Omega^{p,q+1}(J\mathcal{M}_M)$ measures the changes under variations of the metric. As a consequence of $D^2 = 0$ we obtain $d^2 = \delta^2 = d\delta + \delta d = 0$. For example we have $dx^k = Dx^k$, $\delta x^k = 0$, $\delta g_{ij,I} = Dg_{ij,I} - g_{ij,I+k}Dx^k$, $dg_{ij,I} = g_{ij,I+k}Dx^k$.

The diffeomorphisms group of M acts in a natural way on the metrics on M and induces an action on its derivatives. Hence $\mathrm{Diff}M$ acts on $J\mathcal{M}_M$. At the infinitesimal level, for every $X \in \mathfrak{X}(M)$ we obtain a vector field $X_J \in \mathfrak{X}(J\mathcal{M}_M)$.

Accordingly to the splitting (8) the vector field X_J can be expressed as $X_J = H_X + V_X$ where H_X and V_X are the horizontal and vertical components respectively. If in local coordinates $X = X^i\partial/\partial x^i$ then we have

$$H_X = X^i \frac{d}{dx^i}, \tag{9}$$

$$V_X = \delta_X g_{ij}\frac{\partial}{\partial g_{ij}} + \frac{d(\delta_X g_{ij})}{dx^k}\frac{\partial}{\partial g_{ij,k}} + \ldots = \sum_{ij,I}\frac{d^{|I|}(\delta_X g_{ij})}{dx^I}\frac{\partial}{\partial g_{ij,I}}, \tag{10}$$

where the total derivatives are defined by $\frac{d}{dx^k} = \frac{\partial}{\partial x^k} + \sum_{ij,I} g_{ij,I+k}\frac{\partial}{\partial g_{ij,I}}$ and

$$\delta_X g_{ij} = -\left(g_{ij,k}X^k + g_{kj}\partial_i X^k + g_{ik}\partial_j X^k\right).$$

Note that the last expression is similar formula (6) of the previous section and that $\delta_X g_{ij} = \iota_{V_X}\delta g_{ij}$.

If $\alpha \in \Omega^{p,q}(J\mathcal{M}_M)$ then $L_{X_J}\alpha \in \Omega^{p,q}(J\mathcal{M}_M)$ and hence we have

$$L_{X_J}\alpha = \iota_{H_X} d\alpha + d\iota_{H_X}\alpha + \iota_{V_X}\delta\alpha + \delta\iota_{V_X}\alpha,$$
$$\iota_{H_X}\delta\alpha + \delta\iota_{H_X}\alpha = 0,$$
$$\iota_{V_X} d\alpha + d\iota_{V_X}\alpha = 0.$$

The usual constructions in the calculus of variations can be expressed as differential forms in $J\mathcal{M}_M$. For example, if λ is a lagrangian density then $\lambda \in \Omega^{n,0}(J\mathcal{M}_M)$ and $\delta\lambda \in \Omega^{n,1}(J\mathcal{M}_M)$.

We denote by $F^1(J\mathcal{M}_M) \subset \Omega^{n,1}(J\mathcal{M}_M)$ the subspace of forms of the type $A^{ij}\delta g_{ij} \wedge \mathrm{vol}$ (usually this space it is called the space of functional 1-forms or source forms). General results on the variational bicomplex assert that for every $\alpha \in \Omega^{n,1}(J\mathcal{M}_M)$ we have $\alpha = \mathcal{F} - d\theta$, where $\mathcal{F} \in F^1(J\mathcal{M}_M)$ and $\theta \in \Omega^{n-1,1}(J\mathcal{M}_M)$, and the form $\mathcal{F}$ is uniquely determined by α. Moreover, if α is $\mathrm{Diff}M$-invariant then $\mathcal{F}$ and θ can be chosen $\mathrm{Diff}M$-invariant.

In particular, if $\lambda \in \Omega^{n,0}(J\mathcal{M}_M)$ is a lagrangian density then we obtain the first variational formula $\delta\lambda = \mathcal{E} - d\theta$, where $\mathcal{E} = \mathcal{E}^{ij}\delta g_{ij} \wedge \mathrm{vol} \in F^1$ is the Euler-Lagrange form of λ (i.e. $\mathcal{E}^{ij} = 0$ are the Euler-Lagrange equations for λ) and $\theta \in \Omega^{n-1,1}(J\mathcal{M}_M)$ is the symplectic potential. Another example is equation (12) bellow.

If λ is $\mathrm{Diff}M$-invariant we have $0 = L_{X_J}\lambda = \iota_{V_X}\delta\lambda + d\iota_{H_X}\lambda = \iota_{V_X}\mathcal{E} + d\iota_{V_X}\theta + d\iota_{H_X}\lambda$, and hence the conserved current can be defined by $J(X) = \iota_{V_X}\theta + \iota_{H_X}\lambda$ as we have $dJ(X) = -\iota_{V_X}\mathcal{E} \approx 0$.

3.2 Pontryagin forms

At every point of $J\mathcal{M}_M$ the first derivatives can be used to define the Christoffel symbols $\Gamma^i_{jk} = \frac{1}{2}g^{ia}(g_{ak,j} + g_{aj,k} - g_{jk,a})$ and a covariant derivative $\left(D^\Gamma A\right)^i = DA^i + \Gamma^i_{jk}A^j dx^k$ for $A = A^i\partial/\partial x^i \in \Gamma(J\mathcal{M}_M, TM)$. It can be shown that in this way we obtain a well defined connection Γ (called the horizontal Levi -Civita connection) and that Γ is invariant under the natural action of $\mathrm{Diff}M$ (see [9] for details).

Let $\Omega \in \Omega^2(J\mathcal{M}_M, \mathrm{End}TM)$ be the curvature of the connection Γ. Then locally we have

$$\begin{aligned} \Omega^i_j &= D\Gamma^i_{jk} \wedge dx^k + \Gamma^i_{as}\Gamma^a_{jr} dx^s \wedge dx^r \\ &= \delta\Gamma^i_{jk} \wedge dx^k + \tfrac{1}{2}R^i_{jsr} dx^s \wedge dx^r \end{aligned} \tag{11}$$

We write this equation simply by $\Omega = \delta\Gamma + R$. Note that this decomposition corresponds to that in (8). By applying the first Pontryagin polynomial to Ω we obtain the first Pontryagin form $p_1(\Omega) = -\frac{1}{8\pi^2}\mathrm{tr}(\Omega \wedge \Omega) \in \Omega^4(J\mathcal{M}_M)$. For simplicity we set $P = -8\pi^2 p_1(\Omega)$. In dimension 3 if we consider the components of this form, using formula (11) we obtain $P = P_1 + P_2$ with $P_1 = 2\mathrm{tr}(\delta\Gamma \wedge R) \in \Omega^{3,1}(J\mathcal{M}_M)$, $P_2 = \mathrm{tr}(\delta\Gamma \wedge \delta\Gamma) \in \Omega^{2,2}(J\mathcal{M}_M)$.

Equation (4) expressed in terms of forms in the jet bundle gives

$$P_1 = 2\mathrm{tr}(\delta\Gamma \wedge R) = \mathcal{C} - d\eta \tag{12}$$

where

$$\begin{aligned} \mathcal{C} &= C^{ab}\delta g_{ab} \wedge \mathrm{vol} \in F^1 \subset \Omega^{3,1}(J\mathcal{M}_M) \\ \eta &= -2R^i_b \delta g_{ia} \wedge dx^a \wedge dx^b \in \Omega^{2,1}(J\mathcal{M}_M) \end{aligned}$$

In this case both $\mathcal{C}$ and η are $\mathrm{Diff}\, M$-invariant.

3.3 Equivariant Pontryagin forms

We recall the definition of equivariant Pontryagin forms given in [3]. They are used in [10] to study the problem of local gravitational anomaly cancellation, and in [3, 11] are shown to be related to symplectic structures and moment maps in the space of metrics in dimensions $4k-2$. In this paper we show that they are related to conserved currents and black hole entropy of Chern-Simons terms in dimension 3.

We recall that when a connection is invariant under the action of a group G, in addition to the ordinary characteristic classes, we can consider the corresponding G-equivariant characteristics classes, which are closed under the Cartan differential (see [12, 13]). In our case the connection Γ is invariant under the action of $\mathrm{Diff}\, M$ and we can consider the $\mathrm{Diff}\, M$-equivariant Pontryagin forms.

The construction of equivariant Pontryagin forms is based on the following equation (see [3]), which can be obtained directly from equations (9), (10) and (11)

$$\iota_{X_J}\Omega = -D^\Gamma(\nabla X), \tag{13}$$

where $\nabla X^a_b = \partial_b X^a + \Gamma^a_{bc}X^c$. In the decomposition (8) this equation becomes

$$\iota_{V_X}\delta\Gamma + \iota_{H_X}R = -d^\Gamma(\nabla X), \tag{14}$$

$$\iota_{H_X}\delta\Gamma = -\delta(\nabla X). \tag{15}$$

The first equivariant Pontryagin form is defined by (see [3])

$$-\frac{1}{8\pi^2}\mathrm{tr}(\Omega - \nabla X)^2 = -\frac{1}{8\pi^2}\mathrm{tr}(\Omega\wedge\Omega) + \frac{1}{4\pi^2}\mathrm{tr}(\nabla X\cdot\Omega) - \frac{1}{8\pi^2}\mathrm{tr}(\nabla X^2),$$

for $X \in \mathfrak{X}(M)$, and is closed under the Cartan differential $D_C = D - \iota_{X_J}$ by virtue of equation (13). In particular this implies that we have $\iota_{X_J}P = D\left(\beta(X)\right)$, where $\beta(X) = 2\mathrm{tr}(\nabla X\cdot\Omega)$.

A map $\mu\colon \mathfrak{X}(M) \to \Omega^k(J\mathcal{M}_M)$ is $\mathrm{Diff}\, M$-equivariant if $L_{Y_J}(\mu(X)) = \mu([Y,X])$ for every $X, Y \in \mathfrak{X}(M)$. For example, if $\alpha \in \Omega^{k+1}(J\mathcal{M}_M)$ is $\mathrm{Diff}\, M$-invariant, then the map $X \mapsto \iota_{X_J}\alpha$ is equivariant. One of the properties of equivariant Pontryagin forms is that the form $\beta\colon \mathfrak{X}(M) \to \Omega^2(J\mathcal{M}_M)$ is $\mathrm{Diff}\, M$-equivariant.

According to decomposition (8) we have $\beta(X) = \beta_0(X) + \beta_1(X)$, where

$$\beta_0(X) = 2\mathrm{tr}(\nabla X\cdot R) \in \Omega^{2,0}(J\mathcal{M}_M), \tag{16}$$

$$\beta_1(X) = 2\mathrm{tr}(\nabla X\cdot\delta\Gamma) \in \Omega^{1,1}(J\mathcal{M}_M). \tag{17}$$

We show below that β_0 appears on the computation of the conserved current and β_1 appears on the computation of the hamiltonian current and the black hole entropy.

Under the decomposition (8) into horizontal and vertical terms equation $\iota_{X_J} P = D\left(\beta(X)\right)$ becomes

$$\iota_{V_X} P_1 = d\beta_0(X), \tag{18}$$

$$\iota_{H_X} P_1 + \iota_{V_X} P_2 = \delta\beta_0(X) + d\beta_1(X), \tag{19}$$

$$\iota_{H_X} P_2 = \delta\beta_1(X). \tag{20}$$

3.4 Chern-Simons forms

As commented before, the expression (1) for the Chern-Simons lagrangian is only valid locally. However, we show that the Cotton tensor admits a global lagrangian which can be constructed by fixing a metric.

Let $\overline{g} \in \mathrm{Met}M$ be a fixed metric on M, let $\overline{\Gamma}$ be its Levi-Civita connection and $\overline{R}$ its curvature. Then by applying the usual transgression formula we obtain $\mathrm{tr}\left(\Omega^2\right) - \mathrm{tr}(\overline{R}^2) = D\overline{CS}$, where

$$\overline{CS} = 2\mathrm{tr}\left(\overline{a} \wedge \overline{R}\right) + \mathrm{tr}\left(\overline{a} \wedge D^{\overline{\Gamma}}\overline{a}\right) + \frac{2}{3}\mathrm{tr}\left(\overline{a}^3\right).$$

and $\overline{a} = \Gamma - \overline{\Gamma}$. However, $\mathrm{tr}(\overline{R}^2) = 0$ because it is a horizontal 4-form, and hence $P = \mathrm{tr}\left(\Omega^2\right) = D\overline{CS}$.

Accordingly with the decomposition (8) we have $\overline{CS} = \overline{CS}_0 + \overline{CS}_1$ where the second term is $\overline{CS}_1 = \mathrm{tr}(\overline{a} \wedge \delta\Gamma) \in \Omega^{2,1}(J\mathcal{M}_M)$, and the first term $\overline{\lambda}_{CS} = \overline{CS}_0 \in \Omega^{3,0}(J\mathcal{M}_M)$ is a lagrangian density

$$\overline{\lambda}_{CS} = 2\mathrm{tr}\left(\overline{a} \wedge \overline{R}\right) + \mathrm{tr}\left(\overline{a} \wedge d^{\overline{\Gamma}}\overline{a}\right) + \frac{2}{3}\mathrm{tr}\left(\overline{a}^3\right)$$

If in a local chart we choose $\overline{g}$ a constant metric then $\overline{\lambda}_{CS} = \mathrm{tr}\left(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma^3\right)$ is the usual expression of the Chern-Simons lagrangian. Hence $\overline{\lambda}_{CS}$ is a globally well defined lagrangian density with generalizes (1).

The equation $P = D\overline{CS}$ expressed in terms of the decomposition (8) gives

$$P_1 = \delta\overline{CS}_0 + d\overline{CS}_1, \tag{21}$$

$$P_2 = \delta\overline{CS}_1. \tag{22}$$

Obviously $\overline{CS}$ is not $\mathrm{Diff}M$-invariant as it depends on the metric $\overline{g}$. In fact we have

$$L_{X_J}\overline{CS} = \iota_{X_J} D\overline{CS} + D\iota_{X_J}\overline{CS} = \iota_{X_J} P + D\iota_{X_J}\overline{CS} = D(\beta(X) + \iota_{X_J}\overline{CS}).$$

In the decomposition (8) this equation becomes

$$L_{X_J}\overline{\lambda}_{CS} = d(\beta_0(X) + \iota_{V_X}\overline{CS}_1 + \iota_{H_X}\overline{\lambda}_{CS}). \tag{23}$$

$$L_{X_J}\overline{CS}_1 = \delta(\beta_0(X) + \iota_{H_X}\overline{\lambda}_{CS} + \iota_{V_X}\overline{CS}_1) + d(\beta_1(X) + \iota_{H_X}\overline{CS}_1) \tag{24}$$

By equation (12) we have $P_1 = \mathcal{C} - d\eta$. Using this equation and equation (21) we obtain the first variational formula for $\overline{\lambda}_{CS}$

$$\delta\overline{\lambda}_{CS} = P_1 - d\overline{CS}_1 = \mathcal{C} - d\left(\eta + \overline{CS}_1\right) = \mathcal{C} - d\overline{\theta}, \tag{25}$$

where we define the symplectic potential by $\overline{\theta} = \eta + \overline{CS}_1$ (this equation generalizes formula (5)). In particular the Cotton form $\mathcal{C}$ is the Euler-Lagrange form of the Chern-Simons lagrangian. Hence $\overline{\lambda}_{CS}$ is a globally defined lagrangian density, whose Euler-Lagrange operator is the Cotton tensor.

4 Conserved current for the Cotton tensor

The construction of the conserved current for the Chern-Simons term is based on equations (12) and (16). By combining these equations we obtain

$$d\beta_0 = \iota_{V_X} P_1 = 2\delta_X g_{ab} C^{ab}\mathrm{vol} + d\iota_{V_X}\eta.$$

If we define

$$J_{CS}(X) = \alpha(\iota_{V_X}\eta - \beta_0(X))$$

then we have $dJ_{CS}(X) = -2\alpha\delta_X g_{ab}C^{ab}\mathrm{vol} \approx 0$ and hence $J_{CS}(X)$ is a conserved current. Moreover, the invariance of η and the $\mathrm{Diff}\,M$-equivariance of β_0 imply that the map $J_{CS}\colon \mathfrak{X}(M) \to \Omega^{2,0}(J\mathcal{M}_M)$ is $\mathrm{Diff}\,M$-equivariant. An alternative way to obtain the same current is by combining equations (23) and (25).

Next we compute the explicit expression of the conserved current using the results of the preceding sections. We have

$$\iota_{V_X}\eta = -2R^i_c\left(g_{ja}\nabla_i X^j + g_{ji}\nabla_a X^j\right)dx^a \wedge dx^c \tag{26}$$

Moreover, using the formula (3) in the expression (16) of $\beta_0(X)$ we obtain

$$\beta_0(X) = 2\left(\left(g_{ji}\nabla_a X^j - g_{ja}\nabla_i X^j\right)R^i_c - \frac{1}{2}R\nabla_a X^j g_{jc}\right)dx^a \wedge dx^c \tag{27}$$

Finally using (26) and (27) we obtain the expression for the conserved current $J_{CS}(X)$ given by

$$J_{CS}(X) = -4\alpha\nabla_a X^j(R_{jb} - \frac{1}{4}Rg_{jb})dx^a \wedge dx^b \tag{28}$$

It is well known that the conserved currents corresponding to $\mathrm{Diff}\,M$ invariance are globally exact on shell (e.g. see [14]), i.e. we have $J_{CS}(X) \approx dQ_{CS}(X)$ for every $X \in \mathfrak{X}(M)$ where $Q_{CS}(X)$ is called the Noether charge. In our case, if we define the Noether charge by

$$Q_{CS}(X) = -4\alpha X^i(R_{ij} - \frac{1}{4}Rg_{ij})dx^j \tag{29}$$

then, a direct computations shows that we have

$$dQ_{CS}(X) = J_{CS}(X) - 4\alpha g_{ka}C^{ia}X^k\mathrm{vol}_i \approx J_{CS}(X), \tag{30}$$

where $\mathrm{vol}_i = \iota_{\partial_i}\mathrm{vol}$. Moreover, the map $Q\colon \mathfrak{X}(M) \to \Omega^{1,0}(J\mathcal{M}_M)$ is $\mathrm{Diff}M$-equivariant. The tensor $S_{ij} = R_{ij} - \frac{R}{4}g_{ij}$ which appears in the expression of the Noether potential is called the Schouten tensor.

5 Black hole entropy in the presence of Chern-Simons terms

In this section we follow Wald's Noether charge approach to compute the black hole entropy corresponding to the Chern-Simons term in 3 dimensions by using the results of the previous sections. Our approach is similar to that in [5] but we use the geometrical constructions of the previous sections. As it is shown below, in the computation of the black hole entropy the Noether charge disappears, but it appears an additional term related to the form $\beta_1(X)$ given by the equivariant Pontryagin form.

First we recall the basic ideas of Wald's Noether charge method for computing black hole entropy for a $\mathrm{Diff}M$-invariant lagrangian. Let $\lambda \in \Omega^{n,0}(J\mathcal{M}_M)$ be a $\mathrm{Diff}M$ invariant lagrangian density and suppose that we have the first variational formula $\delta\lambda = \mathcal{E} - d\theta$, where $\mathcal{E} \in \Omega^{n,1}(J\mathcal{M}_M)$ is the Euler-Lagrange form of λ and $\theta \in \Omega^{n-1,1}(J\mathcal{M}_M)$ is the symplectic potential, and both of them are $\mathrm{Diff}M$-invariant. The conserved current is given by $J(X) = \iota_{V_X}\theta + \iota_{H_X}\lambda$. Moreover, we have $J(X) \approx dQ(X)$ where $Q(X)$ is the Noether charge.

The form $\omega = \delta\theta \in \Omega^{n-1,2}(J\mathcal{M}_M)$ determines a presymplectic structure on the space of extremals by setting

$$\sigma_g(Y, Z) = \int_\Xi jg^*(\iota_{Z_J}\iota_{Y_J}\omega)$$

where Ξ is a Cauchy hypersurface, g is an extremal metric, and Y, Z are Jacobi fields on g (i.e. variations of the metric satisfying the linearized equations $L_Y\mathcal{E}|g = 0$) and if locally we have $Y = y_{ij}dx^i dx^j$ then Y_J is the vector field on $J\mathcal{M}_M$ given by $Y_J = \sum_{ij,I} \frac{\partial^{|I|}y_{ij}}{\partial x^I}\frac{\partial}{\partial g_{ij,I}}$.

Using the invariance of θ we obtain

$$\iota_{V_X}\omega \simeq d(\delta Q(X) - \iota_{H_X}\theta), \tag{31}$$

where $\simeq$ means modulo terms that vanish when the form is contracted with a Jacobi field and evaluated in an extremal metric. Hence we have

$$\iota_{\delta_X g}\sigma = \int_\Xi d(\delta Q(X) - \iota_{H_X}\theta) = \delta\int_{\partial\Xi} Q(X) - \int_{\partial\Xi}\iota_{H_X}\theta \tag{32}$$

If $\partial\Xi$ is an asymptotic $(n-2)$-sphere at infinity Σ_∞ and $\iota_{H_X}\theta = \delta B(X)$ for certain $B(X) \in \Omega^{n-2,0}(J\mathcal{M}_M)$ then $H(X) = \int_{\Sigma_\infty}(Q(X) - B(X))$ can be considered as the Hamiltonian function corresponding to the vector field X, as it satisfies Hamilton's equation $\iota_{\delta_X g}\sigma = \delta H(X)$.

Now let us suppose that we have a stationary black hole spacetime with a bifurcate Killing horizon Σ generated by ξ, and let Ξ be an asymptotically flat

surface having Σ as its only interior boundary. As ξ is a Killing vector field we have $\delta_\xi g = 0$ and the right hand side of equation (32) vanishes. If $\xi = \partial_t + \Omega\partial_\phi$ where ∂_t is the generator of the global time translation, Ω the angular velocity of the horizon and ∂_ϕ the angular rotation then we obtain

$$\delta \int_\Sigma Q(\xi) = \delta \int_{\Sigma_\infty} Q(\partial_t) + \Omega\delta \int_{\Sigma_\infty} Q(\partial_\phi) - \int_{\Sigma_\infty} \iota_{H_{\partial_t}}\theta$$

where Σ_∞ is the asymptotic infinity of the Cauchy surface, and we have used that $\xi|_\Sigma = 0$ and hence $\iota_{H_\xi}\theta|_\Sigma = 0$, and that $\iota_{H_{\partial_\phi}}\theta|_{\Sigma_\infty} = 0$ because ∂_ϕ is tangent to Σ_∞.

By defining the energy by $\mathcal{E} = \int_{\Sigma_\infty}(Q(\partial_t) - B(\partial_t))$, the angular momentum by $\mathcal{J} = -\int_{\Sigma_\infty} Q(\partial_\phi)$ and the entropy by $S = \frac{2\pi}{\kappa}\int_\Sigma Q(\xi)$ we obtain the first law of black hole thermodynamics $\kappa\delta S = \delta\mathcal{E} - \Omega\delta\mathcal{J}$ where κ is the surface gravity of the black hole characterized by $\nabla\xi|_\Sigma = \kappa\epsilon$ and ϵ is the binormal to Σ.

For a Diff$M$-invariant lagrangian Wald's formula gives a general expression (see [14, 15, 16])

$$S = -2\pi \int_\Sigma \frac{\delta L}{\delta R_{abcd}} \epsilon_{ab}\varepsilon_{cdi_1\ldots i_{d-2}} dx^{i_1}\ldots dx^{i_{d-2}}$$

where $\lambda = L\mathrm{vol}$ and L is considered as a function of the curvature tensor R and its covariant derivatives.

In $3D$ topologically massive gravity the lagrangian includes a Chern-Simons term which is not Diff$M$-invariant, and hence Wald's formula cannot be applied in this case. We follow Wald's Noether charge approach to obtain a formula for the entropy and we show that our result coincides with that obtained in [5].

For the Chern-Simons lagragian we have the first variational formula (25). Hence in our case the presymplectic structure on the space of solutions σ is determined by the form $\alpha\omega$, where

$$\omega = \delta\overline{\theta} = \delta\eta + \delta\overline{CS}_1 = \delta\eta + P_2.$$

Note that although $\overline{\theta}$ depends on the metric $\overline{g}$ and hence is not Diff$M$-invariant, the form ω is Diff$M$-invariant. As η is Diff$M$-invariant we have

$$L_{X_J}\eta = \iota_{H_X} d\eta + d\iota_{H_X}\eta + \delta\iota_{V_X}\eta + \iota_{V_X}\delta\eta = 0.$$

Using this equation we obtain

$$\iota_{V_X}\omega = \iota_{V_X}\delta\eta + \iota_{V_X}P_2 = -\iota_{H_X}d\eta - d\iota_{H_X}\eta - \delta\iota_{V_X}\eta + \iota_{V_X}P_2 \tag{33}$$

By equations (12) and (19) we have

$$\iota_{H_X}d\eta = \iota_{H_X}\mathcal{C} - \iota_{H_X}P_1 \simeq -\delta\beta_0(X) - d\beta_1(X) + \iota_{V_X}P_2$$

Moreover, by equation (30) we have

$$\iota_{V_X}\eta - \beta_0(X) = J_{CS}(X) = dQ_{CS}(X) + 2g_{ka}\mathcal{C}^{ia}X^k\mathrm{vol}_i \simeq dQ_{CS}(X).$$

Replacing this equations on (33) we obtain

$$\iota_{V_X}\omega \simeq d\left(\delta Q_{CS}(X) + \beta_1(X) - \iota_{H_X}\eta\right) \tag{34}$$

Finally, using equation (15) we have

$$\beta_1(X) = 2\mathrm{tr}(\nabla X \cdot \delta\Gamma) = \delta\left(2\mathrm{tr}(\nabla X \cdot \overline{a}) - \mathrm{tr}(\iota_{H_X}\overline{a}\cdot\overline{a})\right) + \iota_{H_X}\mathrm{tr}(\overline{a}\wedge\delta\Gamma)$$

By replacing the last equation on (34) we obtain the analogous of equation (31) for the Chern-Simons lagrangian

$$\iota_{V_X}\omega \simeq d\left(\delta q_{CS}(X) + \iota_{H_X}\nu\right). \tag{35}$$

where we define

$$\begin{aligned} q_{CS}(X) = Q_{CS}(X) + 2\mathrm{tr}(\nabla X \cdot \overline{a}) - \mathrm{tr}(\iota_{H_X}\overline{a}\cdot\overline{a}), \\ \nu = \mathrm{tr}(\overline{a}\wedge\delta\Gamma) - \eta \end{aligned}$$

Hence for the Chern-Simons lagrangian the hamiltonian current q_{CS} plays the same role as Q for $\mathrm{Diff}\,M$-invariant lagrangians in the computation of black hole entropy.

Choosing $\overline{g}$ a constant metric we obtain $q(X) = Q(X) + \mathrm{tr}(2\nabla X \cdot \Gamma) - \mathrm{tr}(\iota_{H_X}\Gamma\cdot\Gamma)$. In local coordinates we have

$$q_{CS}(X) = \left(-4S_{ir}X^i dx^r + 2\partial_j X^i \Gamma^j_{ir} + \Gamma^i_{jk}\Gamma^j_{ir}X^k\right)dx^r$$

where S_{ij} are the components of the Schouten tensor.

This expression is similar to the current defined in [4], given by

$$q_2(X) = \left(-4S_{ir}X^i + \Gamma^i_{jk}\Gamma^j_{ir}X^k\right)dx^r$$

Note that the difference between q_{CS} and q_2 are the terms containing derivatives of X, and hence for a constant vector both expressions coincide. For example in [4] $q_2(X)$ is used to compute the contribution of the Chern-Simons term to the energy (or mass) and angular momentum for BTZ black holes, Log-gravity and Warped AdS_3 black holes by setting $\mathcal{E}_{CS} = \alpha\int_{\Sigma_\infty} q_2(\partial_t)$ and $\mathcal{J}_{CS} = \alpha\int_{\Sigma_\infty} q_2(\partial_\phi)$. If the vector fields ∂_t and ∂_ϕ are constant we have $\mathcal{E}_{CS} = \alpha\int_{\Sigma_\infty} q_{CS}(\partial_t)$ and $\mathcal{J}_{CS} = \alpha\int_{\Sigma_\infty} q_{CS}(\partial_\phi)$.

Moreover, the current q_{CS} also gives the correct result for the entropy. If $\xi = \partial_t + \Omega\partial_\phi$, over Σ we have $\xi|_\Sigma = 0$ and hence $\iota_{H_\xi}\nu = 0$ and $q(\xi)|_\Sigma = 2\mathrm{tr}(\nabla\xi\cdot\Gamma)|_\Sigma = 2\kappa\mathrm{tr}(\epsilon\cdot\Gamma)$. Hence we can define the contribution of the Chern-Simons term to the black hole entropy by

$$S_{CS} = \frac{2\pi\alpha}{\kappa}\int_\Sigma q(\xi) = 4\pi\alpha\int_\Sigma \mathrm{tr}(\epsilon\cdot\Gamma).$$

This expression coincides with the result obtained in [5, §3.1]. In [5] this expression is computed for the BTZ black hole and shown to coincide with the results obtained by other methods in [17, 18, 19, 20, 21, 22].

As commented in the Introduction, most of our geometrical constructions can be extended to higher dimensions. However, the analogous of equation (35) needed to define the black hole entropy is not so simple in dimensions greater than 3.

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